Solution to HW6

Leon Li

Academic Building 1, Room 505

ylli@math.cuhk.edu.hk

§16.1

9. Evaluate $\int_C (x+y)\,ds$ where C is the straight-line segment $x = t, y = (1-t), z = 0$, from $(0, 1, 0)$ to $(1, 0, 0)$.

Sol) $\vec{r}(t) = t\hat{i} + (1-t)\hat{j}$; $\vec{r}\,'(t) = \hat{i} - \hat{j}$; $\|\vec{r}\,'(t)\| = \sqrt{1^2+1^2+0^2} = \sqrt{2}$

$$\therefore \int_C (x+y)\,ds = \int_0^1 (t + (1-t)) \cdot |\vec{r}\,'(t)|\,dt$$

$$= \int_0^1 \sqrt{2}\,dt = \sqrt{2}.$$

11. Evaluate $\int_C (xy + y + z)\,ds$ along the curve $\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + (2-2t)\mathbf{k}$, $0 \le t \le 1$.

Sol) $\vec{r}(t) = 2t\hat{i} + t\hat{j} + (2-2t)\hat{k}$; $\vec{r}\,'(t) = 2\hat{i} + \hat{j} - 2\hat{k}$; $|\vec{r}\,'(t)| = \sqrt{2^2+1^2+(-2)^2} = 3$

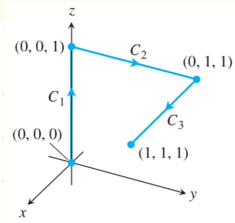
$$\therefore \int_C (xy+y+z)\,ds = \int_0^1 (2t\cdot t + t + 2 - 2t) \cdot |\vec{r}\,'(t)|\,dt$$

$$= \int_0^1 (2t^2 - t + 2) \cdot 3\,dt$$

$$= 3\left[\frac{2t^3}{3} - \frac{t^2}{2} + 2t\right]_0^1 = 3 \cdot \left(\frac{2}{3} - \frac{1}{2} + 2\right) = \frac{13}{2} //$$

16. Integrate $f(x, y, z) = x + \sqrt{y} - z^2$ over the path from $(0, 0, 0)$ to $(1, 1, 1)$ (see accompanying figure) given by

$$C_1: \quad \mathbf{r}(t) = t\mathbf{k}, \quad 0 \le t \le 1$$
$$C_2: \quad \mathbf{r}(t) = t\mathbf{j} + \mathbf{k}, \quad 0 \le t \le 1$$
$$C_3: \quad \mathbf{r}(t) = t\mathbf{i} + \mathbf{j} + \mathbf{k}, \quad 0 \le t \le 1$$



Sol) $C_1 : \vec{r}(t) = t\hat{k}\ ;\ \vec{r}\,'(t) = \hat{k}\ ;\ |\vec{r}\,'(t)| = \sqrt{0^2+0^2+1^2} = 1$

$$\therefore \int_{C_1} (x+\sqrt{y}-z^2)\,ds = \int_0^1 (0+0-t^2)\cdot|\vec{r}\,'(t)|\,dt = \int_0^1 (-t^2)\,dt = \left[-\frac{t^3}{3}\right]_0^1 = -\frac{1}{3}.$$

$C_2 : \vec{r}(t) = t\hat{j}+\hat{k}\ ;\ \vec{r}\,'(t) = \hat{j}\ ;\ |\vec{r}\,'(t)| = \sqrt{0^2+1^2+0^2} = 1$

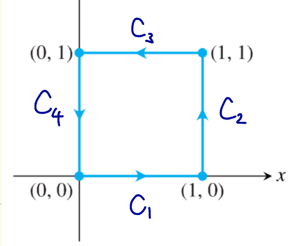
$$\therefore \int_{C_2} (x+\sqrt{y}-z^2)\,ds = \int_0^1 (0+\sqrt{t}-1^2)\cdot|\vec{r}\,'(t)|\,dt = \int_0^1 (\sqrt{t}-1)\,dt = \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} - t\right]_0^1 = -\frac{1}{3}$$

$C_3 : \vec{r}(t) = t\hat{i}+\hat{j}+\hat{k}\ ;\ \vec{r}\,'(t) = \hat{i}\ ;\ |\vec{r}\,'(t)| = \sqrt{1^2+0^2+0^2} = 1$

$$\therefore \int_{C_3} (x+\sqrt{y}-z^2)\,ds = \int_0^1 (t+\sqrt{1}-1^2)\cdot|\vec{r}\,'(t)|\,dt = \int_0^1 t\,dt = \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2}$$

$$\therefore \int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \int_{C_3} f\,ds = -\frac{1}{3} - \frac{1}{3} + \frac{1}{2} = -\frac{1}{6}.$$

26. Evaluate $\displaystyle\int_C \frac{1}{x^2+y^2+1}\,ds$ where C is given in the accompanying figure.



Sol) $C_1 : \vec{r}(t) = t\hat{i}$, $0 \leq t \leq 1$; $\vec{r}'(t) = \hat{i}$; $|\vec{r}'(t)| = \sqrt{1^2+0^2} = 1$

$$\therefore \int_{C_1} \frac{1}{x^2+y^2+1}\,ds = \int_0^1 \frac{1}{t^2+0^2+1}\cdot|\vec{r}'(t)|\,dt = \int_0^1 \frac{1}{t^2+1}\,dt = [\tan^{-1}(t)]_0^1 = \frac{\pi}{4}.$$

$C_2 : \vec{r}(t) = \hat{i}+t\hat{j}$, $0 \leq t \leq 1$; $\vec{r}'(t) = \hat{j}$; $|\vec{r}'(t)| = \sqrt{0^2+1^2} = 1$

$$\therefore \int_{C} \frac{1}{x^2+y^2+1}\,ds = \int_0^1 \frac{1}{1^2+t^2+1}\cdot|\vec{r}'(t)|\,dt = \int_0^1 \frac{1}{t^2+2}\,dt = \left[\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)\right]_0^1 = \frac{\sqrt{2}}{2}\cdot\tan^{-1}\left(\frac{1}{\sqrt{2}}\right).$$

$C_3 : \vec{r}(t) = (1-t)\hat{i}+\hat{j}$, $0 \leq t \leq 1$; $\vec{r}'(t) = -\hat{i}$; $|\vec{r}'(t)| = \sqrt{(-1)^2+0^2} = 1$

$$\therefore \int_{C_3} \frac{1}{x^2+y^2+1}\,ds = \int_0^1 \frac{1}{(1-t)^2+1^2+1}\cdot|\vec{r}'(t)|\,dt = \int_0^1 \frac{1}{(t-1)^2+2}\,dt = \left[\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{t-1}{\sqrt{2}}\right)\right]_0^1 = \frac{\sqrt{2}}{2}\cdot\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$C_4 : \vec{r}(t) = (1-t)\hat{j}$, $0 \leq t \leq 1$; $\vec{r}'(t) = -\hat{j}$; $|\vec{r}'(t)| = \sqrt{0^2+(-1)^2} = 1$

$$\therefore \int_{C_4} \frac{1}{x^2+y^2+1}\,ds = \int_0^1 \frac{1}{0^2+(1-t)^2+1}\cdot|\vec{r}'(t)|\,dt = \int_0^1 \frac{1}{(t-1)^2+1}\,dt = [\tan^{-1}(t-1)]_0^1 = \frac{\pi}{4}$$

$$\therefore \int_C \frac{1}{x^2+y^2+1}\,ds = \int_{C_1} \frac{1}{x^2+y^2+1}\,ds + \int_{C_2} \frac{1}{x^2+y^2+1}\,ds + \int_{C_3} \frac{1}{x^2+y^2+1}\,ds + \int_{C_4} \frac{1}{x^2+y^2+1}\,ds$$

$$= \frac{\pi}{4} + \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) + \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) + \frac{\pi}{4}$$

$$= \frac{\pi}{2} + \sqrt{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) //$$

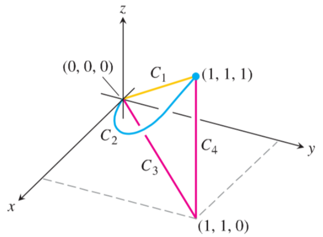
§16.2

In Exercises 7–12, find the line integrals of **F** from (0, 0, 0) to (1, 1, 1) over each of the following paths in the accompanying figure.

a. The straight-line path C_1: $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 1$

b. The curved path C_2: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k}, \quad 0 \le t \le 1$

c. The path $C_3 \cup C_4$ consisting of the line segment from (0, 0, 0) to (1, 1, 0) followed by the segment from (1, 1, 0) to (1, 1, 1)



9. $\mathbf{F} = \sqrt{z}\mathbf{i} - 2x\mathbf{j} + \sqrt{y}\mathbf{k}$

Sol) $\vec{F}(x,y,z) = \sqrt{z}\,\hat{i} - 2x\,\hat{j} + \sqrt{y}\,\hat{k}$

(a) C_1: $\vec{r}(t) = t\hat{i} + t\hat{j} + t\hat{k}$; $\vec{r}'(t) = \hat{i} + \hat{j} + \hat{k}$; $(\vec{F}\cdot\vec{r}')(t) = \sqrt{t} - 2t + \sqrt{t} = 2(\sqrt{t} - t)$

$$\therefore \int_{C_1} (\vec{F}\cdot\vec{r}')(t)\,dt = \int_0^1 2(\sqrt{t} - t)\,dt = 2\left[\frac{t^{3/2}}{3/2} - \frac{t^2}{2}\right]_0^1 = \frac{1}{3}$$

(b) C_2: $\vec{r}(t) = t\hat{i} + t^2\hat{j} + t^4\hat{k}$; $\vec{r}'(t) = \hat{i} + 2t\hat{j} + 4t^3\hat{k}$; $(\vec{F}\cdot\vec{r}')(t) = t^2 - 2t\cdot(2t) + t\cdot 4t^3$

$= 4t^4 - 3t^2$

$$\therefore \int_{C_2} (\vec{F}\cdot\vec{r}')(t)\,dt = \int_0^1 4t^4 - 3t^2\,dt = \left[\frac{4t^5}{5} - t^3\right]_0^1 = -\frac{1}{5}$$

(c) C_3: $\vec{r}(t) = t\hat{i} + t\hat{j}$, $0 \le t \le 1$; $\vec{r}'(t) = \hat{i} + \hat{j}$;

$(\vec{F}\cdot\vec{r}')(t) = 0 - 2t + 0 = -2t \quad \therefore \int_{C_3} (\vec{F}\cdot\vec{r}')(t)\,dt = \int_0^1 (-2t)\,dt = [-t^2]_0^1 = -1$

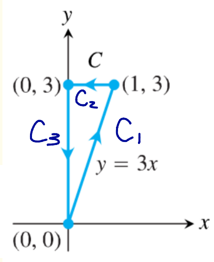
C_4: $\vec{r}(t) = \hat{i} + \hat{j} + t\hat{k}$, $0 \le t \le 1$; $\vec{r}'(t) = \hat{k}$

$(\vec{F}\cdot\vec{r}')(t) = 0 + 0 + 1 = 1$, $\therefore \int_{C_4} (\vec{F}\cdot\vec{r}')(t)\,dt = \int_0^1 dt = [t]_0^1 = 1$

$$\therefore \int_{C_3 \cup C_4} (\vec{F}\cdot\vec{r}')(t)\,dt = \int_{C_3} (\vec{F}\cdot\vec{r}')(t)\,dt + \int_{C_4} (\vec{F}\cdot\vec{r}')(t)\,dt = -1 + 1 = 0$$

In Exercises 13–16, find the line integrals along the given path C.

16. $\int_C \sqrt{x + y}\, dx$, where C is given in the accompanying figure.



Sol) $C_1 : \vec{r}(t) = t\hat{i} + 3t\hat{j}$, $0 \le t \le 1$; $\vec{r}\,'(t) = \hat{i} + 3\hat{j}$, $dx = dt$

$$\therefore \int_{C_1} \sqrt{x+y}\, dx = \int_0^1 \sqrt{t+3t}\, dt = \int_0^1 2\sqrt{t}\, dt = 2\left[\frac{t^{3/2}}{3/2}\right]_0^1 = \frac{4}{3}$$

$C_2 : \vec{r}(t) = (1-t)\hat{i} + 3\hat{j}$. $0 \le t \le 1$; $\vec{r}\,'(t) = -\hat{i}$, $dx = -dt$

$$\int_{C_2} \sqrt{x+y}\, dx = \int_0^1 \sqrt{1-t+3}\,(-dt) = -\int_0^1 \sqrt{4-t}\, dt = \left[\frac{-(4-t)^{3/2}}{3/2}\right]_0^1 = \frac{2}{3}(3\sqrt{3} - 8)$$

$C_3 : \vec{r}(t) = -t\hat{j}$, $0 \le t \le 3$; $\vec{r}\,'(t) = -\hat{j}$, $dx = 0$

$$\int_{C_3} \sqrt{x+y}\, dx = \int_0^3 \sqrt{0+3-t} \cdot 0 = 0$$

$$\therefore \int_C \sqrt{x+y}\, dx = \int_{C_1} \sqrt{x+y}\, dx + \int_{C_2} \sqrt{x+y}\, dx + \int_{C_3} \sqrt{x+y}\, dx$$

$$= \frac{4}{3} + \frac{2}{3}(3\sqrt{3} - 8) + 0$$

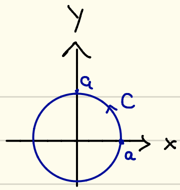
$$= 2\sqrt{3} - 4 \;//$$

30. Flux across a circle Find the flux of the fields

$$\mathbf{F}_1 = 2x\mathbf{i} - 3y\mathbf{j} \qquad \text{and} \qquad \mathbf{F}_2 = 2x\mathbf{i} + (x - y)\mathbf{j}$$

across the circle

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}, \qquad 0 \le t \le 2\pi.$$



Sol) $\vec{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j}$; $\vec{r}\,'(t) = -a\sin t\,\hat{i} + a\cos t\,\hat{j}$

$\vec{n}\,|\vec{v}| = a\cos t\,\hat{i} + a\sin t\,\hat{j}$

$\vec{F}_1(\vec{r}(t)) = 2a\cos t\,\hat{i} - 3a\sin t\,\hat{j}$; $\vec{F}_1 \cdot \vec{n}\,|V| = 2a^2\cos^2 t - 3a^2\sin^2 t$

$\therefore$ Flux of $\vec{F}_1 = \int_0^{2\pi} (2a^2\cos^2 t - 3a^2\sin^2 t)\,dt$

$= a^2\left[\left(t + \frac{\sin 2t}{2}\right) - \frac{3}{2}\left(t - \frac{\sin 2t}{2}\right)\right]_0^{2\pi} = -\pi a^2$ //

$\vec{F}_2(\vec{r}(t)) = 2a\cos t\,\hat{i} + (a\cos t - a\sin t)\hat{j}$

$\vec{F}_2 \cdot \vec{n}\,|V| = 2a^2\cos^2 t + (a\sin t)(a\cos t - a\sin t)$

$= a^2(2\cos^2 t + \sin t\cos t - \sin^2 t)$

Flux of $\vec{F}_2 = \int_0^{2\pi} a^2(2\cos^2 t + \sin t\cos t - \sin^2 t)\,dt$

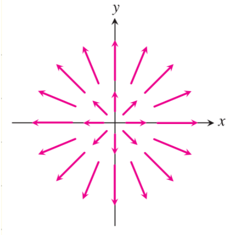
$= \int_0^{2\pi} a^2(2\cos^2 t + \sin t\cos t - \sin^2 t)\,dt$

$= a^2\left[\left(t + \frac{\sin 2t}{2}\right) + \frac{\sin^2 t}{2} - \left(\frac{t}{2} - \frac{\sin 2t}{4}\right)\right]_0^{2\pi} = \pi a^2$ //

40. Radial field Draw the radial field

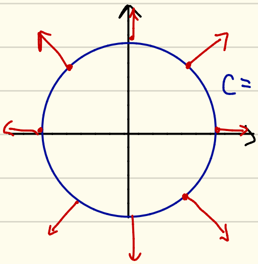
$$\mathbf{F} = x\mathbf{i} + y\mathbf{j}$$

(see Figure 16.11) along with its horizontal and vertical components at a representative assortment of points on the circle $x^2 + y^2 = 1$.



Sol)

Representatives (x,y)	$\vec{F}(x,y)$
$(1,0)$	$\hat{i}$
$(-1,0)$	$-\hat{i}$
$(0,1)$	$\hat{j}$
$(0,-1)$	$-\hat{j}$
$(\frac{1}{2}, \frac{\sqrt{3}}{2})$	$\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}$
$(-\frac{1}{2}, \frac{\sqrt{3}}{2})$	$-\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}$
$(\frac{1}{2}, -\frac{\sqrt{3}}{2})$	$\frac{1}{2}\hat{i} - \frac{\sqrt{3}}{2}\hat{j}$
$(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$	$-\frac{1}{2}\hat{i} - \frac{\sqrt{3}}{2}\hat{j}$



$C = \{(x,y) \mid x^2+y^2=1\}$

44. Two "central" fields Find a field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in the xy-plane with the property that at each point $(x, y) \neq (0, 0)$, $\mathbf{F}$ points toward the origin and $|\mathbf{F}|$ is **(a)** the distance from (x, y) to the origin, **(b)** inversely proportional to the distance from (x, y) to the origin. (The field is undefined at $(0, 0)$.)

Sol) Note that $-\frac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}}$ points toward the origin with magnitude 1.

(a) $\vec{F}(x,y) = \sqrt{x^2+y^2}\left(-\frac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}}\right) = -x\hat{i} - y\hat{j}$

(a) $\vec{F}(x,y) = \frac{C}{\sqrt{x^2+y^2}}\left(-\frac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}}\right) = -C\left(\frac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}}\right)$, where $C \neq 0$.

46. Work done by a radial force with constant magnitude A particle moves along the smooth curve $y = f(x)$ from $(a, f(a))$ to $(b, f(b))$. The force moving the particle has constant magnitude k and always points away from the origin. Show that the work done by the force is

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = k\left[(b^2 + (f(b))^2)^{1/2} - (a^2 + (f(a))^2)^{1/2}\right].$$

Sol) $\vec{r}(x) = x\hat{i} + f(x)\hat{j}$; $\vec{r}\,'(x) = \hat{i} + f'(x)\hat{j}$;

$$\vec{F}(x,y) = \frac{k}{\sqrt{x^2+y^2}}(x\hat{i} + y\hat{j}) \; ; \; \vec{F}(\vec{r}(x)) = \frac{k}{\sqrt{x^2+(f(x))^2}}(x\hat{i} + f(x)\hat{j})$$

$$\vec{F}(\vec{r}(x)) \cdot \vec{r}\,'(x) = \frac{k}{\sqrt{x^2+(f(x))^2}}(x + f'(x) \cdot f(x)) = k\frac{d}{dx}\left(\sqrt{x^2+(f(x))^2}\right)$$

$$\therefore \int_C \vec{F} \cdot \vec{T}\, ds = \int_a^b k \frac{d}{dx}\left(\sqrt{x^2+(f(x))^2}\right) dx$$

$$= k\left[\sqrt{x^2+(f(x))^2}\right]_a^b = k\left(\sqrt{b^2+(f(b))^2} - \sqrt{a^2+(f(a))^2}\right) //$$